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Duality and the geometry of quantum mechanics

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Abstract

A recent notion in theoretical physics is that not all quantum theories arise from quantizing a classical system. Also, a given quantum model may possess more than just one classical limit. There is strong evidence for these facts in string duality and M-theory, and it has been suggested that they should also have a counterpart in quantum mechanics. In view of these developments we propose *dequantization*, a mechanism to render a quantum theory classical. Specifically, we present a geometric procedure to *dequantize* a given quantum mechanics (regardless of its classical origin, if any) to possibly different classical limits, whose quantization gives back the original quantum theory. The standard classical limit $\hbar \to 0$ arises as a particular case of our approach.

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1. Introduction

1.1. Motivation

Approaching quantum mechanics from a geometric viewpoint is a very interesting topic. The goal is a geometrization of quantum mechanics [1], similar in spirit to that of classical mechanics [2, 3]. Beyond this similarity, however, there are numerous deep reasons. One of these is motivated in string duality and M-theory [4, 5]. In plain words, we are confronted with the fact that not all quantum theories arise from quantizing a classical system. Also, a given quantum model may possess more than just one classical limit. These two facts are in sharp contrast with our current understanding of quantum mechanics. While it is true that these two phenomena originally arise in the theories of strings and branes [6], some authors [5] have expressed the opinion that they should somehow be reflected at the fundamental level of quantum mechanics as well. Let us describe the general set-up.

Quantization may be understood as a prescription to construct a quantum theory from a given classical theory. As such, it is far from being unique. Beyond canonical quantization and functional integrals, a number of different, often complementary approaches to quantization are known, each one exploiting different aspects of the underlying classical theory. For example, geometric quantization [7–10] relies on the geometric properties of classical mechanics. Systems whose classical phase space $\mathcal{C}$ is a Kähler manifold can be quantized as in [11–13]. If $\mathcal{C}$ is just a Poisson manifold, then the approach of [14], based on deformation quantization [15,16], can always be applied. A path-integral counterpart to these mathematical techniques has been developed in [17].

A common feature to these approaches is the fact that they all take a classical mechanics as a starting point. Thus the classical limit is *a fortiori* unique: it reduces to letting $\hbar \to 0$. If we want to allow for the existence of more than one classical limit, we are led to considering a quantum mechanics that is not based, at least primarily, on the the quantization of a given classical dynamics. In such an approach one would not take first a classical model and then quantize it. Rather, quantum mechanics itself would be the starting point: a parent quantum theory would give rise, in a certain limit, to a classical theory. If there are several different ways of taking this limit, then there will be several different classical limits.

1.2. Summary

In this paper we put forward a geometric proposal by which quantum mechanics can be rendered classical, or *dequantized*, in more than one way, thus yielding different classical limits. Under *dequantization* we understand the following.

Assume that classical phase space $\mathcal{C}$ is $\boldsymbol{R}^{2n}$. Then, starting from the quantum phase space $\mathcal{Q}$ of standard quantum mechanics [1], the usual classical limit $\hbar \to 0$ is obtained as the quotient of $\mathcal{Q}$ by a certain equivalence relation $\sim$, i.e. $\mathcal{Q}/\sim = \boldsymbol{R}^{2n}$, and we have a trivial fibre bundle $\mathcal{Q} \to \boldsymbol{R}^{2n}$. We will construct classical phase spaces $\mathcal{Q}/G = \mathcal{C}$, where G is a Lie group acting on $\mathcal{Q}$, and $\mathcal{Q} \to \mathcal{C}$ will be a (not necessarily trivial) G-bundle. The associated vector bundle will have $\mathcal{H}$, the Hilbert space of quantum states, as its typical fibre. In order to qualify as a classical phase space, $\mathcal{C}$ must be a symplectic manifold whose quantization must give back the original quantum theory on $\mathcal{Q}$. Different choices for G will give rise to different classical limits.

1.3. Outline

This paper is organized as follows. Section 2 summarizes the standard Hilbert space formulation of quantum mechanics, following the geometric presentation of [1]. We will recall how the standard classical limit $\hbar \to 0$ is taken. In this analysis, a natural mechanism will arise that will allow more than one classical limit to exist. This is presented in section 3. We illustrate our technique with some specific examples in section 4, where one given quantum mechanics is explicitly *dequantized*. The physical implications of our proposal are discussed in section 5. Some technical mathematical aspects of our construction are elucidated in section 6.

2. A geometric approach to quantum mechanics

For later purposes let us briefly summarize the geometric approach to quantum mechanics presented in [1]. Throughout this section our use of the terms *classical* and *quantum* will be the standard one [18].

2.1. The Hilbert space as a Kähler manifold

The starting point is an infinite-dimensional, complex, separable Hilbert space of quantum states, $\mathcal{H}$, that is most conveniently viewed as a real vector space equipped with a complex structure J. Correspondingly, the Hermitian inner product can be decomposed into real and imaginary parts,

$$\langle\phi, \psi\rangle = g(\phi, \psi) + \mathrm{i}\omega(\phi, \psi), \tag{1}$$

with g a positive-definite, real scalar product and ω a symplectic form. The metric g, the symplectic form ω and the complex structure J are related as

$$g(\phi, \psi) = \omega(\phi, J\psi), \tag{2}$$

which means that the triple (J, g, ω) endows the Hilbert space $\mathcal{H}$ with the structure of a Kähler space [2].

Thus any Hilbert space naturally gives rise to a symplectic manifold: it is the *quantum* phase space $\mathcal{Q}$, or the space of rays in $\mathcal{H}$. Let $\omega_{\mathcal{Q}}$ denote the restriction of ω to $\mathcal{Q}$. On $\mathcal{Q}$, the inverse of $\omega_{\mathcal{Q}}$ can be used to define Poisson brackets and Hamiltonian vector fields. This is done as follows.

Any function $f_{\mathcal{C}} : \mathcal{C} \to \boldsymbol{R}$ defined on classical phase space $\mathcal{C}$ has associated a self-adjoint quantum observable F on $\mathcal{H}$. The latter gives rise to a quantum function $f_{\mathcal{Q}} : \mathcal{Q} \to \boldsymbol{R}$ on quantum phase space $\mathcal{Q}$, defined as the expectation value of the operator F:

$$f_{\mathcal{Q}}(\psi) = \langle\psi, F\psi\rangle. \tag{3}$$

Now every function $f : \mathcal{Q} \to \boldsymbol{R}$ defines a Hamiltonian vector field X_f through the equation [3]

$$\mathrm{i}_{X_f}\omega_{\mathcal{Q}} = \mathrm{d}f. \tag{4}$$

In this way the Poisson bracket $\{\,,\,\}_{\mathcal{Q}}$ on $\mathcal{Q}$ is defined by [3]

$$\{f_{\mathcal{Q}}, g_{\mathcal{Q}}\}_{\mathcal{Q}} = \omega_{\mathcal{Q}}(X_f, X_g). \tag{5}$$

Let us now consider the classical coordinate and momentum functions $q^j_{\mathcal{C}}$ and $p^k_{\mathcal{C}}$ satisfying the canonical Poisson brackets on $\mathcal{C}$. Through the above construction one arrives at the quantum coordinate and momentum functions $q^j_{\mathcal{Q}}$ and $p^k_{\mathcal{Q}}$ satisfying the canonical Poisson brackets on $\mathcal{Q}$:

$$\{q^j_{\mathcal{Q}}, p^k_{\mathcal{Q}}\}_{\mathcal{Q}} = \delta^{jk}, \qquad \{q^j_{\mathcal{Q}}, q^k_{\mathcal{Q}}\}_{\mathcal{Q}} = 0 = \{p^j_{\mathcal{Q}}, p^k_{\mathcal{Q}}\}_{\mathcal{Q}}. \tag{6}$$

It turns out that Hamilton's canonical equations of motion on $\mathcal{Q}$ are equivalent to Schrödinger's wave equation, while the Riemannian metric g accounts for properties such as the measurement process and Heisenberg's uncertainty relations.

We are thus dealing with two phase spaces, which we denote $\mathcal{C}$ (for *classical*) and $\mathcal{Q}$ (for *quantum*). $\mathcal{Q}$ is always infinite-dimensional, as it derives from an infinite-dimensional Hilbert space. In contrast, $\mathcal{C}$ may well be finite-dimensional. Furthermore, while both $\mathcal{C}$ and $\mathcal{Q}$ are symplectic manifolds, the latter is always Kähler, while the former need not be Kähler.

Two questions arise naturally. Firstly, what is the geometric relation between $\mathcal{C}$ and $\mathcal{Q}$ as manifolds? Secondly, how are $\mathcal{C}$ and $\mathcal{Q}$ related as *symplectic* manifolds, i.e., how are their respective symplectic forms $\omega_{\mathcal{C}}$ and $\omega_{\mathcal{Q}}$ related? When $\mathcal{C} = \boldsymbol{R}^{2n}$, the answer is provided in [1] and summarized below.

2.2. Quantum phase space as a fibre bundle over classical phase space

For a classical system with n degrees of freedom, let us collectively denote by $f_r, r = 1, \ldots, 2n$, the quantum coordinate and momentum functions $q^j_{\mathcal{Q}}$ and $p^k_{\mathcal{Q}}$. We define an equivalence relation on $\mathcal{Q}$ as

$$x_1 \sim x_2 \qquad \text{iff } f_r(x_1) = f_r(x_2)\, \forall r. \tag{7}$$

Through this equivalence relation, the quantum phase space $\mathcal{Q}$ becomes a trivial fibre bundle with fibre $\mathcal{H}$ over the classical phase space $\boldsymbol{R}^{2n}$:

$$\mathcal{Q} \longrightarrow \mathcal{Q}/\sim\, = \boldsymbol{R}^{2n}. \tag{8}$$

2.3. Relation between the classical and the quantum symplectic forms

A tangent vector $v \in T_x\mathcal{Q}$ is said to be vertical at $x \in \mathcal{Q}$ if $v(f_r) = 0\ \forall r$. Therefore the vertical directions are those in which the quantum coordinate and momentum functions assume constant values. Equivalently, the vertical subspace $\mathcal{V}_x$ at $x \in \mathcal{Q}$ may be defined as

$$\mathcal{V}_x = \{v \in T_x\mathcal{Q} : \omega_{\mathcal{Q}}(X_{f_r}(x), v) = 0\ \forall r\}. \tag{9}$$

Let $\mathcal{V}_x^{\perp}$ denote the $\omega_{\mathcal{Q}}$-orthogonal complement of the vertical subspace at $x \in \mathcal{Q}$. Each tangent space splits as the direct sum

$$T_x\mathcal{Q} = \mathcal{V}_x \oplus \mathcal{V}_x^{\perp}, \tag{10}$$

and the tangent vectors that lie in $\mathcal{V}_x^{\perp}$ are said to be horizontal at x. It turns out that the quantum states lying on a horizontal cross section of the bundle (8) are precisely the generalized coherent states of [19, 20].

Now, if u and v are vectors on $\mathcal{C} = \boldsymbol{R}^{2n}$, we denote by u^h and v^h their horizontal lifts to $\mathcal{Q}$. Then the classical symplectic structure $\omega_{\mathcal{C}}$ is related to its quantum counterpart $\omega_{\mathcal{Q}}$ through

$$\omega_{\mathcal{C}}(u, v) = \omega_{\mathcal{Q}}(u^h, v^h), \tag{11}$$

i.e., $\omega_{\mathcal{C}}$ is the horizontal part of $\omega_{\mathcal{Q}}$.

3. Taking a classical limit

The geometric presentation summarized in section 2 makes it clear that the quantum theory contains all the information about the classical theory. In this sense, as explained in section 1, we should think of quantum mechanics as being prior to classical mechanics. Rather than quantizing a classical theory, rendering quantum mechanics classical, or *dequantizing* it, appears to be the key issue. How can one *dequantize*?

3.1. Symplectic reduction

Our primary concern will be to obtain a classical *symplectic* manifold $(\mathcal{C}, \omega_{\mathcal{C}})$ from its quantum counterpart $(\mathcal{Q}, \omega_{\mathcal{Q}})$, in such a way that the quantization of $(\mathcal{C}, \omega_{\mathcal{C}})$ will reproduce $(\mathcal{Q}, \omega_{\mathcal{Q}})$ as a *symplectic* manifold, regardless of the Riemannian metric $g_{\mathcal{C}}$ on $\mathcal{C}$, if any. In principle, *dequantization* may be thought of as the symplectic reduction from $(\mathcal{Q}, \omega_{\mathcal{Q}})$ to a symplectic submanifold $(\mathcal{C}, \omega_{\mathcal{C}})$; a more general definition will be given in section 3.3. In having $\mathcal{C}$ as a reduced symplectic manifold of $\mathcal{Q}$ we are assured that the quantization of $\mathcal{C}$ reproduces $\mathcal{Q}$. See [3, 21] for a treatment of symplectic reduction.

We do not require the metric $g_{\mathcal{Q}}$ on $\mathcal{Q}$ to descend to a metric $g_{\mathcal{C}}$ on $\mathcal{C}$. Disregarding the metric $g_{\mathcal{C}}$ is justified, as the metric $g_{\mathcal{Q}}$ of equation (1) can always be obtained from the symplectic form $\omega_{\mathcal{Q}}$ through the Kähler condition (2).

In contrast, the symplectic structure is an essential ingredient to keep in the passage from quantum to classical, as classical phase space is always symplectic. In what follows we will consider symplectic structures as in [22, 23] but, more generally, one could relax $\mathcal{C}$ to be a Poisson manifold.

3.2. Reduction via fibre bundles

A useful approach to symplectic reduction is via fibre bundles. When $\mathcal{C} = \boldsymbol{R}^{2n}$, the classical limit arises in [1] as the base space of a trivial fibre bundle with fibre $\mathcal{H}$ and total space $\mathcal{Q}$. This suggests considering fibre bundles $\mathcal{Q} \to \mathcal{C}$, with fibre $\mathcal{H}$ and total space $\mathcal{Q}$, over some other finite-dimensional base manifold $\mathcal{C}$. If the classical phase space $\mathcal{C}$ so obtained is a symplectic manifold whose quantization reproduces the initial quantum theory on $\mathcal{Q}$, then associated with that fibre bundle there is one classical limit.

Let us first examine trivial fibre bundles. The equivalence relation of section 2.2 is singled out because it is well suited to obtain the standard coherent states of [19, 20]. We will see in section 4.3 one particular example of a certain group G acting on $\mathcal{Q}$ such that $\mathcal{Q}/G = \mathcal{C}$ coincides with the result of taking the standard classical limit $\hbar \to 0$. The procedure of section 4.3 is in fact quite general in order to replace equivalence relations with group actions.

Nontrivial fibre bundles may also be considered. They provide a realization of the statement presented in [24], to theeffect that one can always choose local coordinates on classical phase space, in terms of which quantization becomes a local expansion in powers of $\hbar$ around a certain local vacuum. This expansion is local in nature: it does not hold globally on classical phase space when the fibre bundle is nontrivial. In this sense, quantization is mathematically reminiscent of the local triviality property satisfied by every fibre bundle [25] while, physically, it is reminiscent of the equivalence principle of general relativity [26].

3.3. Definition of dequantization

For our purposes, *dequantization* will mean the following. Let G be a Lie group acting on $\mathcal{Q}$. Moding out by the action of G we will construct principal G-bundles

$$\mathcal{Q} \longrightarrow \mathcal{Q}/G = \mathcal{C} \tag{12}$$

over finite-dimensional symplectic manifolds $\mathcal{C}$. We require the associated vector bundle to have $\mathcal{H}$ as its fibre. Moreover the lift of $\omega_{\mathcal{C}}$ to $\mathcal{Q}$ must equal $\omega_{\mathcal{Q}}$.

Equation (11) expressed the property that, when $\mathcal{C} = \boldsymbol{R}^{2n}$, $\omega_{\mathcal{C}}$ was simply the horizontal part of $\omega_{\mathcal{Q}}$. Horizontality was closely related to coherence. Here we have no notion of horizontality because *any* $\omega_{\mathcal{C}}$ will work, provided its lift to $\mathcal{Q}$ equals $\omega_{\mathcal{Q}}$ (as is the case, for example, in symplectic reduction). In general, the best we can do is to find local canonical coordinates on $\mathcal{C}$ in terms of which

$$\omega_{\mathcal{C}} = \mathrm{d}p_k \wedge \mathrm{d}q^k. \tag{13}$$

With respect to these local coordinates, local coherent states $|z_k\rangle$ can be defined simply as eigenvectors of the local annihilation operator $a_k = Q^k + \mathrm{i}P_k$, where Q^k and P_k are the quantum observables corresponding to q^k and p_k. How do Q^k and P_k *dequantize* to q^k and p_k?

3.4. Classical functions from quantum observables

When *dequantizing*, instead of having classical functions $f_{\mathcal{C}} : \mathcal{C} \to \boldsymbol{R}$ to turn into quantum observables F, we have quantum observables F out of which we would like to obtain classical functions. We can use equation (3) in order to define the quantum function $f_{\mathcal{Q}} : \mathcal{Q} \to \boldsymbol{R}$ corresponding to the observable F. Now, in the examples that follow, $\mathcal{C}$ is a submanifold of $\mathcal{Q}$. Hence the restriction of $f_{\mathcal{Q}}$ to $\mathcal{C}$ gives rise to a well-defined classical function $f_{\mathcal{C}} : \mathcal{C} \to \boldsymbol{R}$ whose quantization reproduces the quantum observable F.

4. Examples of different classical limits

In the following we give some examples of the *dequantization* of the nonrelativistic quantum mechanics of n degrees of freedom. We will concentrate on some specific nonlinear choices for the manifold $\mathcal{C}$, namely complex projective spaces $\boldsymbol{CP}^n$ and complex submanifolds thereof. Linear classical phase spaces have been dealt with in sections 2.2 and 2.3. Coherent states on spheres have been constructed in [27].

4.1. The standard coherent states

Points in $\boldsymbol{CP}^n$ may be specified by homogeneous coordinates $[w_0 : \ldots : w_n]$ on $\boldsymbol{C}^{n+1}$. Alternatively, holomorphic coordinates on $\boldsymbol{CP}^n$ in the chart with, say, $w_0 \neq 0$, are given by $z_k = w_k/w_0$, with $k = 1, \ldots, n$.

In order to discuss coherent states it is convenient to use homogeneous coordinates. Then we have a Kähler form

$$\omega = \mathrm{i} \sum_{k=0}^{n} \mathrm{d}w^k \wedge \mathrm{d}\bar{w}^k, \tag{14}$$

which we take to define a symplectic structure with invariance group $U(n+1)$. As we are working in homogeneous coordinates we still have to mod out by $U(1)$, so the true invariance group of the Kähler form is $G = U(n+1)/U(1) \simeq SU(n+1)$. Let $G' \subset G$ be a maximal isotropy subgroup [20] of the vacuum state $|0\rangle$. Coherent states $|\zeta\rangle$ are parametrized by points ζ in the coset space G/G' [20]. Set $n = 1$ for simplicity, so $\boldsymbol{CP}^1 \simeq S^2$. Then $G' = U(1)$, and coherent states $|u\rangle$ are parametrized by points u in the quotient space $S^2 = SU(2)/U(1)$.

We will find it convenient to recall Berezin's quantization [11] of the Riemann sphere. The Hilbert space is most easily presented in holomorphic coordinates $z, \bar{z}$, which have the advantage of being almost global coordinates on S^2. The Kähler potential

$$K_{S^2}(z, \bar{z}) = \log(1 + |z|^2) \tag{15}$$

produces an integration measure

$$\mathrm{d}\mu\,(z, \bar{z}) = \frac{1}{2\pi\mathrm{i}} \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1 + |z|^2)^2}. \tag{16}$$

The Hilbert space of states is the space $\mathcal{F}_\hbar(S^2)$ of holomorphic functions on S^2 with finite norm, the scalar product being

$$\langle\psi_1|\psi_2\rangle = \left(\frac{1}{\hbar} + 1\right) \int_{S^2} \mathrm{d}\mu\,(z, \bar{z})\,(1 + |z|^2)^{-1/\hbar}\,\overline{\psi}_1(z)\,\psi_2(z). \tag{17}$$

It turns out that $\hbar^{-1}$ must be an integer. For ψ to have finite norm, it must be a polynomial of degree less than $\hbar^{-1}$. In fact, setting $\hbar^{-1} = 2j + 2$, $\mathcal{F}_\hbar(S^2)$ is the representation space for the spin-j representation of $SU(2)$, which is the isometry group of S^2. The semiclassical regime corresponds to $j \to \infty$.

4.2. Interlude: the unitary group of Hilbert space

Some care is needed when dealing with the unitary group of Hilbert space $\mathcal{H}$. In what follows we will consider the groups $U(\mathcal{H})$ and $U(\infty)$, whose definitions and properties we examine next.

We first recall the following theorem [25]: a sufficient condition for a fibre bundle to be trivial is that either the stucture group or the base manifold be contractible to a point. Hence the classical limit may be nonglobal only if both the structure group and the base manifold of the fibre bundle are noncontractible.

By definition, the group $U(\mathcal{H})$ comprises all unitary transformations of $\mathcal{H}$. One may topologize $U(\mathcal{H})$ with different, nonequivalent topologies. Two popular choices are the norm topology and the strong operator topology [28]. However, it turns out that both of them render $U(\mathcal{H})$ contractible [28, 29], so

$$\pi_1(U(\mathcal{H})) = 0 \tag{18}$$

and all principal $U(\mathcal{H})$-bundles are trivial, whatever the base manifold.

Next let us consider the group $U(n)$ of $n \times n$ unitary matrices u in a fixed, finite dimension n. We may think of u as acting on $\mathcal{H}$, which is infinite-dimensional, by simply enlarging it with an infinite-dimensional identity matrix:

$$\begin{pmatrix} u & \cdot \\ \cdot & \mathbf{1} \end{pmatrix}. \tag{19}$$

In this way we have the inclusions $U(n) \subset U(\mathcal{H})$ for all n. Now enlarge every $n \times n$ unitary matrix to an $(n+1) \times (n+1)$ unitary matrix by adding one row and one column. The group $U(\infty)$ is defined by performing this enlargement infinitely many times. This we denote as

$$U(\infty) = \lim_{n\to\infty} U(n). \tag{20}$$

Defined in this way, $U(\infty)$ is a strict subgroup of $U(\mathcal{H})$,

$$U(\infty) \subset U(\mathcal{H}), \tag{21}$$

because *not* every element of $U(\mathcal{H})$ can be obtained in the manner just described.

For $U(\infty)$ to become a topological group, we need to endow it with a topology. However, on $U(\infty)$ we do not want to consider the topology induced by $U(\mathcal{H})$. For reasons that will become clear presently, we want to topologize $U(\infty)$ with a topology of its own. In order to do this we consider the usual topology on $U(n)$ (the one induced by $\boldsymbol{C}^n$) and ask ourselves, what topology is there on $U(\infty)$ that renders every matrix inclusion

$$U(n) \subset U(\infty) \tag{22}$$

continuous. The answer to this question is known in the mathematical literature [30]: there exists on $U(\infty)$ a topology, called the *direct limit topology*, that renders every matrix inclusion (22) continuous, while at the same time being the *maximal* topology that enjoys this property. In this way $U(\infty)$ becomes a topological group. Moreover, this topology also respects the fundamental group

$$\pi_1(U(n)) = \boldsymbol{Z} \tag{23}$$

in the passage $n \to \infty$:

$$\pi_1(U(\infty)) = \boldsymbol{Z}. \tag{24}$$

Hence $U(\infty)$ is *not* contractible to a point, and principal $U(\infty)$-bundles over noncontractible base manifolds may be nontrivial.

4.3. Global coherent states from trivial fibre bundles

Let $\mathcal{Q}$ be the manifold of rays in $\mathcal{H}$. We define an action of the group $U(\infty)$ on $\mathcal{Q}$ as follows: first lift $\mathcal{Q}$ to $\mathcal{H}$, then apply a $U(\infty)$ transformation. In this way we obtain a fibre bundle whose base is $\mathcal{C} = \mathcal{Q}/U(\infty)$. Now any two points in $\mathcal{Q}$ can always be connected by means of a transformation in $U(\infty)$, so this $\mathcal{C}$ reduces to a point. This is an instance of the situation mentioned in section 3, that *not* every bundle will give rise to a reasonable classical limit.

A sensible classical limit is the following. $\mathcal{H}$ being infinite-dimensional, we may require the action of $U(\infty)$ to act as the identity along, say, the first $n+1$ complex dimensions of $\mathcal{H}$, while allowing it to act nontrivially on the rest. In this way the resulting $\mathcal{C} = \mathcal{Q}/U(\infty)$ is the complex n-dimensional projective space $\boldsymbol{CP}^n$. It is the base of a principal fibre bundle whose total space is $\mathcal{Q}$ and whose fibre is $U(\infty)$. This bundle is trivial by construction.

Next consider the trivial vector bundle, with fibre $\mathcal{H}$, that is associated with this trivial principal bundle. Triviality implies that one has the globally defined diffeomorphism $\mathcal{Q} \simeq \mathcal{C} \times \mathcal{H}$. Now coherent states lie on sections of this bundle. Hence the triviality of this bundle ensures that these coherent states are globally defined on $\mathcal{C}$. An equivalent phrasing of this statement is to say that the semiclassical regime is globally defined on $\mathcal{C}$. Upon quantization, all observers on $\mathcal{C}$ will agree on what is a semiclassical versus what is a strong quantum effect. Setting $n = 1$ for simplicity, if one observer on $\mathcal{C}$ measures $j < \infty$, then so will all other observers. If the measure is $j \to \infty$, then so will it be for all other observers, too.

Now $U(\infty)$ is the invariance group of the Kähler form on $\mathcal{Q}$

$$\omega = \mathrm{i}\sum_{k=0}^{\infty} \mathrm{d}w^k \wedge \mathrm{d}\bar{w}^k. \tag{25}$$

The Kähler form on the resulting $\boldsymbol{CP}^n$ is given in equation (14), i.e., it is the one obtained by quotienting (25) with this group action. Incidentally, the metric g on $\mathcal{Q}$ also descends to the quotient $\boldsymbol{CP}^n$, and we can now apply Berezin's quantization [11]. In fact we have picked our group action precisely so as to obtain a *dequantization* of $\mathcal{Q}$ to $\boldsymbol{CP}^n$ that exactly reproduces the standard classical limit $\hbar \to 0$ for $\boldsymbol{CP}^n$. Similarly, the corresponding coherent-state quantization [19] is the one summarized in section 4.1. This example also illustrates the power of fibrating $\mathcal{Q}$ by means of a group action. Yet another choice for the group action will lead to another different *dequantization*.

4.4. Local coherent states from nontrivial fibre bundles

Let us consider the Hopf bundle

$$S^{2n+1}/U(1) \simeq \boldsymbol{CP}^n, \tag{26}$$

where the sphere S^{2n+1} is the submanifold of $\boldsymbol{C}^{n+1}$ defined by

$$|z_0|^2 + \cdots + |z_n|^2 = 1, \tag{27}$$

and the $U(1)$ action is

$$(z_0, \ldots, z_n) \mapsto \mathrm{e}^{\mathrm{i}\alpha}(z_0, \ldots, z_n). \tag{28}$$

This fibre bundle is nontrivial [31].

We define an action of $U(\infty)$ on $\mathcal{Q}$ as follows: first lift $\mathcal{Q}$ to the infinite-dimensional sphere S^∞, then embed S^∞ into $\mathcal{H}$ using equation (27) in the limit $n \to \infty$, then apply a $U(\infty)$ transformation. We require that this action be given by equation (28) on the first $n+1$ dimensions of $\mathcal{H}$, i.e., only a $U(1)$ subgroup of $U(\infty)$ will act on them. Along the remaining infinite dimensions we let $U(\infty)$ act unconstrained. In this way we obtain a principal $U(\infty)$

fibre bundle whose base $\mathcal{C}$ is $\boldsymbol{CP}^n$ and whose total space is $\mathcal{Q}$. This $\boldsymbol{CP}^n$ inherits its symplectic structure (14) by quotienting (25) with the group action, so its standard quantization reproduces the original quantum theory on $\mathcal{Q}$, up to an important difference. Coherent states (regarded as sections of the associated vector bundle whose fibre is $\mathcal{H}$) are no longer globally defined on $\boldsymbol{CP}^n$ because this bundle is nontrivial by construction, and therefore it admits no global section.

The physical implications of the local character of these coherent states are easy to interpret. Again set $n = 1$ for simplicity. In the case of the trivial bundle of section 4.3, the cross section of coherent states above *any* observer on the base $\boldsymbol{CP}^1$ was globally defined. Hence the semiclassical regime was universally defined for all observers on $\boldsymbol{CP}^1$. In contrast, the nontriviality of the bundle considered here implies that the semiclassical regime is defined only locally, and it *cannot* be extended globally over $\boldsymbol{CP}^1$. What to one observer appears to be a semiclassical effect need *not* appear so to a different observer.

4.5. Submanifolds of complex projective space

Any smooth, complex algebraic manifold M given by a system of polynomial equations in $\boldsymbol{CP}^n$ has a natural symplectic structure [2]. Let $\iota : M \to \boldsymbol{CP}^n$ be an embedding of the complex manifold M into complex projective space. Then the symplectic form ω on $\boldsymbol{CP}^n$ can be pulled back to a symplectic form $\iota^*\omega$ on M. The fibre bundles of sections 4.3 and 4.4, when pulled back to M, naturally suggest new instances of classical limits. The submanifold M must satisfy the integrality conditions [10].

5. Physical discussion

We would first of all like to emphasize that *dequantization* has already been proposed in the literature; see [32–35] for some important works on this topic. These papers present interesting geometric approaches that explore in detail the deep link existing between classical and quantum mechanics. Moreover, these *dequantizations* also allow for the non-uniqueness of the classical limit advocated here.

The approach presented in this paper, as compared with [32–35], is motivated in duality properties found in strings and M-theory, as summarized in section 1. We have developed a mechanism by which coherent states can be defined only locally on $\mathcal{C}$, instead of globally. In this way, two different observers on $\mathcal{C}$ need not agree on whether a certain given phenomenon is strong quantum or semiclassical. What one of them terms *semiclassical* may well be perceived by the other as *strong quantum*. To the best of our knowledge, this viewpoint is new, even if our techniques and results may partially overlap with those in [32–35].

We have in sections 4.3 and 4.4 presented an example of a group action on separable, infinite-dimensional complex Hilbert space $\mathcal{H}$ that leads to two different dequantizations of the quantum mechanics of a point particle. The coherent states obtained in section 4.3 are those of the standard Berezin quantization of $\boldsymbol{CP}^n$, while the coherent states of section 4.4 may be interpreted as belonging to a nontrivial quotient of the former. In particular, locally on $\mathcal{C}$, the latter coincide with Berezin's coherent states for $\boldsymbol{CP}^n$, but globally they do not.

An interesting extension of our approach would be to formulate a classification theorem for *dequantizations*, analogous to the Kostant–Souriau theorem of geometric quantization [7, 8]. Such an approach requires classifying all possible quotients $\mathcal{Q}/G$ of quantum phase space under G-actions. Work is in progress along this line [36]; we hope to report on this in the near future.

We have emphasized the key role played by the symplectic structure in switching back and forth between $\mathcal{Q}$ and $\mathcal{C}$. In contrast, the role played by the Riemannian metric $g_{\mathcal{C}}$ has been reduced to that of providing quantum numbers once a certain classical limit has been fixed. It is precisely through lifting the metric dependence that we have succeeded in obtaining different classical limits for a given quantum theory. In this sense, as suggested in [24], implementing duality transformations in quantum mechanics is very reminiscent of topological field theory.

Lifting the metric dependence in favour of diffeomorphism invariance, as in topological theories, is also important for the following reasons. We have made no reference to coupling constants or potentials, with the understanding that the Hamilton–Jacobi method has already placed us, by means of suitable coordinate transformations, in a coordinate system where all interactions vanish. At least under the standard notions [18] of *classical* versus *quantum*, this is certainly always possible at the classical level [37]. At the quantum level, the approach of [38], which contains standard quantum mechanics as a limiting case, rests precisely on the possibility of transforming between any two states by means of diffeomorphisms. Diffeomorphism invariance is a very powerful tool. It can be used [38] in the passage from *classical* to *quantum*. It can also be applied in the passage from *quantum* to *quantum*, as in [39], where Hamiltonian quantum theories are constructed from functional integrals in the Osterwalder–Schrader framework [40, 41]. The viewpoint advocated here is that it can also be successfully applied in the passage from *quantum* to *classical*.

Then the only truly *quantum* ingredient we have at hand is $\hbar$. In fact one can think of quantization, especially of deformation quantization [15, 16], as performing an infinite expansion in powers of $\hbar$ around a classical theory. This full infinite expansion gives the full quantum theory. *Dequantization* may then be interpreted as the truncation of this infinite expansion to a given finite order. As we have argued, if the quantum fibre bundle $\mathcal{Q} \to \mathcal{C}$ is nontrivial, this expansion in powers of $\hbar$ is local instead of global, so the notion of *classical* versus *quantum* may not be globally defined for all observers.

6. Mathematical discussion

To conclude we would like to comment on some interesting mathematical points of our construction.

In the construction of the Hopf bundle of section 4.4 one could ask, why not use a $U(1)$ subgroup of $U(\mathcal{H})$ instead of $U(\infty)$? In fact one could do so, but at the cost of rendering the whole infinite-dimensional bundle over $\mathcal{C}$ trivial; only the finite-dimensional subbundle corresponding to the Hopf bundle would remain nontrivial. There would be no contradiction, since the triviality of a given bundle does not prevent the existence of nontrivial subbundles. For example, given any vector bundle $E \to \mathcal{C}$ over a (compact and Hausdorff) base manifold $\mathcal{C}$, there always exists a complementary vector bundle $F \to \mathcal{C}$ such that $E \oplus F$ is trivial [42].

However, the situation just described is precisely what we want to avoid. We need the complete, infinite-dimensional bundle over $\mathcal{C}$ to be nontrivial in order for the classical limit not to be globally defined; a finite-dimensional subbundle will not suffice. In retrospect, this argument also justifies our choice of $U(\infty)$ in section 4.4. The topologies considered above on $U(\mathcal{H})$, while rendering every inclusion $U(n) \subset U(\mathcal{H})$ continuous, are not the *maximal* topology enjoying that property. On the contrary, the direct limit topology on $U(\infty)$ is the maximal one with that property. This ensures that the addition of an infinite number of (spectator) dimensions to the n-dimensional Hopf bundle (26) does *not* render the complete infinite-dimensional bundle trivial, as would be the case with $U(\mathcal{H})$.

Quantum-mechanical symmetries are usually implemented by the action of unitary operators on $\mathcal{H}$. The group $U(\mathcal{H})$ thus arises naturally in this set-up. However, any principal bundle with structure group $U(\mathcal{H})$ is necessarily trivial. In retrospect, this explains why the classical limit is always considered to be globally defined. In order to bypass this difficulty we have considered the subgroup $U(\infty) \subset U(\mathcal{H})$ and endowed it with a topology of its own (the direct limit topology) that is different from the induced topology it would inherit from $U(\mathcal{H})$. Only so do we have a chance of rendering $U(\infty)$-bundles nontrivial. It is interesting to observe that $U(\infty)$, instead of $U(\mathcal{H})$, is the right group that contains all $U(n)$ groups, in a way that naturally respects their topologies. $U(n)$ groups arise naturally in theories with solitons and instantons. In supersymmetric Yang–Mills theories and superstring theory, solitons and instantons lie at the heart of the notion of duality. This supports the notion that implementing duality transformations in quantum mechanics is in fact possible through mechanisms such as the one proposed here. It would also be very interesting to extend our mechanism to more general quantum-mechanical structures such as rigged Hilbert spaces [43].

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References

[1] Ashtekar A and Schilling T 1997 *Preprint* gr-qc/9706069
Ashtekar A 1999 *Preprint* qr-qc/9901023
[2] Arnold V 1989 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)
[3] Marsden J and Ratiu T 1999 *Introduction to Mechanics and Symmetry* (Berlin: Springer)
[4] Schwarz J 1997 *Nucl. Phys.* B **55** 1 (thep-th/0007130, hep-ex/0008017)
[5] Vafa C 1997 *Preprint* hep-th/9702201
[6] Kaku M 1999 *Introduction to Superstrings and M-Theory* (Berlin: Springer)
Kaku M 2000 *Strings, Conformal Fields and M-Theory* (Berlin: Springer)
[7] Kostant B 1970 *Lectures in Modern Analysis and Applications* vol 3, ed C Taam (Berlin: Springer)
Kostant B 1970 *Group Representations in Mathematics and Physics* ed V Bargmann (Berlin: Springer)
[8] Souriau J-M 1976 *Group Theoretical Methods in Physics* ed A Janner, T Janssen and M Boon (Berlin: Springer)
Souriau J-M 1997 *Structure of Dynamical Systems* (Basel: Birkhäuser)
[9] Ŝniatycki 1980 *Geometric Quantization and Quantum Mechanics* (Berlin: Springer)
[10] Woodhouse N 1991 *Geometric Quantization* (Oxford: Oxford University Press)
[11] Berezin F 1974 *Sov. Math. Izv.* **38** 1116
Berezin F 1975 *Sov. Math. Izv.* **39** 363
Berezin F 1975 *Commun. Math. Phys.* **40** 153
Berezin F 1978 *Commun. Math. Phys.* **63** 131
[12] Schlichenmaier M 1996 *Preprint* q-alg/9601016
Schlichenmaier M 1996 *Preprint* q-alg/9611022
Schlichenmaier M 1999 *Preprint* math.QA/9902066
Schlichenmaier M 1999 *Preprint* math.QA/9910137
[13] Reshetikhin N and Takhtajan L 1999 *Preprint* math.QA/9907171
[14] Kontsevich M 1997 *Preprint* q-alg/9709040
Kontsevich M 1999 *Lett. Math. Phys.* **48** 35
[15] For a review see Sternheimer D 1998 *Preprint* math.QA/9809056
[16] Zachos C 2001 *Preprint* hep-th/0110114
[17] Cattaneo A and Felder G 2000 *Commun. Math. Phys.* **212** 591
[18] Galindo A and Pascual P 1990 *Quantum Mechanics* vols I, II (Berlin: Springer)
[19] Klauder J and Skagerstam B-S 1985 *Coherent States* (Singapore: World Scientific)
Gilmore R 1972 *Ann. Phys., NY* **74** 391
[20] Perelomov A 1986 *Generelized Coherent States and their Applications* (Berlin: Springer)
[21] Marsden J and Weinstein A 1974 *Rep. Math. Phys.* **5** 121
[22] Guillemin V and Sternberg S 1984 *Symplectic Techniques in Physics* (Cambridge: Cambridge University Press)
[23] Müller O 2001 *Preprint* math-ph/0108016
[24] Isidro J M 2001 *Preprint* quant-ph/0105012
[25] Steenrod N 1970 *The Topology of Fibre Bundles* (Princeton, NJ: Princeton University Press)

[26] Weinberg S 1972 *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (New York: Wiley)
[27] Hall B and Mitchell J 2001 *Preprint* quant-ph/0109086
[28] Dixmier J 1969 *Les C*-Algèbres et leurs Représentations* (Paris: Gauthier-Villars)
[29] Kuiper N 1965 *Topology* **3** 19
[30] Bourbaki N 1958 *Topologie Générale* (Paris: Hermann)
[31] Schwarz A 1994 *Topology for Physicists* (Berlin: Springer)
[32] Etingof P and Kazhdan D 1998 Quantization of Lie bialgebras, II, III *Sel. Math. (N.S.)* **4** 213–31
Etingof P and Kazhdan D 1998 *Sel. Math. (N.S.)* **4** 233–69
[33] Hennings M, Dubin D and Smith T 1998 Dequantization techniques for Weyl quantization *Publ. Res. Inst. Math. Sci. (Kyoto University)* **34** 325–54
[34] Ali T and Antoine J 1994 *Quantum Frames, Quantization and Dequantization. Collection Quantization and Infinite-Dimensional Systems (Bialowieza, 1993)* (New York: Plenum) pp 133–45
[35] Berceanu S and Gheorghe A 1991 Classical motion and quantum evolution on compact homogeneous complex manifolds of coherent states *Differential Geometry and Topology Applications in Physics and Technics (Bucharest)*
Berceanu S and Gheorghe A 1991 *Polytech. Inst. Bucharest Sci. Bull. Electr. Eng.* **53** 51–60
[36] Isidro J M 2002 in preparation
[37] Dittrich W and Reuter M *Classical and Quantum Dynamics* (Berlin: Springer)
[38] Faraggi A and Matone M 2000 *Int. J. Mod. Phys.* A **15** 1869
Bertoldi G, Faraggi A and Matone M 2000 *Class. Quantum Grav.* **17** 3965
[39] Ashtekar A, Marolf D, Mourão J and Thiemann T 2000 *Class. Quantum Grav.* **17** 4919
[40] Osterwalder K and Schrader R 1973 *Commun. Math. Phys.* **31** 83
Osterwalder K and Schrader R 1975 *Commun. Math. Phys.* **42** 281
[41] Glimm J and Jaffe A 1980 *Quantum Physics* (Berlin: Springer)
[42] Atiyah M 1967 *K-Theory* (New York: Benjamin)
[43] De la Madrid R, Bohm A and Gadella M 2001 *Preprint* quant-ph/0109154